Math 259A Lecture 21 Notes

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1 Amenable Groups and Algebras

1.1 Equivalence of amenability for groups and algebras

Definition 1.1. A II_1 factor M (with a trace state τ) has **property Gamma** if for all $x_1, \dots, x_n \in M$ and $\varepsilon > 0$, there exists some $u \in U(M)$ such that $\tau(u) = 0$ and $\|u x_i u^* - x_i\|_\tau < \varepsilon$ for all i.

Here $\|x\|_\tau = \tau(x^* x)^{1/2}$. Last time, we showed the folowing:

Theorem 1.1. *1. $L(S_\infty)$ has property Gamma.*

2. For $n \geq 2$, $L(\mathbb{F}_2)$ does not.

Corollary 1.1. $L(S_\infty) \not\cong L(\mathbb{F}_n)$.

Definition 1.2. Γ is **amenable** if it has an **invariant mean** (i.e. a state $\varphi \in S(\ell^\infty(\Gamma))$ such that $\varphi(gf) = \varphi(f)$ for all $f \, in \ell^\infty$ and $g \in \Gamma$.

Definition 1.3. Γ satisfies Følner's condition if for all nonempty, finite $F \subseteq \Gamma$, for every $\varepsilon > 0$, ther eis a finite $K \subseteq \Gamma$ such that $\frac{|FK \triangle K|}{|K|} < \varepsilon$.

Theorem 1.2. *Γ satisfies Følner's condition if and only if it has an invariant mean.*

We only did the ($\implies$) direction, but we will do the other direction later.

Definition 1.4. A II_1 factor (M, τ) is **amenable** if there exists a state $\varphi \in S(\mathcal{B}(L^2(M, \tau)))$ satisfying $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and $T \in \mathcal{B}(L^2(M))$ (here, $L^2(M) := \overline{M}^{\|\cdot\|_\tau}$). This is equivalent to $\varphi(uTu^*) = \varphi(T)$ for all $u \in U(M)$ and $T \in \mathcal{B}(L^2(M))$. Such a φ is called a **hypertrace**, as $\varphi|_M = \tau$.

In the case where $M = L(\Gamma)$, this is $\varphi \in S(\mathcal{B}(\ell^2(\Gamma)))$.

Theorem 1.3. *Let Γ be an ICC group. $M = L(\Gamma)$ is amenable if and only if Γ is amenable.*

Proof. ($\Longleftarrow$): Take the trace τ on M and extend it to a state on $\mathcal{B}(L^2(M)) = \mathcal{B}(\ell^2(\Gamma))$: take $\widetilde{\tau}(T) = \langle T\xi_e, \xi_e\rangle$ for all $T \in \mathcal{B}(\ell^2(\Gamma))$. Let ψ be an invariant mean on Γ. Define

$$\varphi(T) = \int_\Gamma \widetilde{\tau}(u_g T u_g^*)\, d\psi.$$

By this integration, we mean $\psi((\widetilde{\tau}(u_g T u_g^*))_g)$. Then $\varphi(T) = \varphi(u_h T u_h^*$ for all $h \in \Gamma$, φ is linear, and $\varphi(1) = 1$. So φ is a state on $\mathcal{B}(L^2)$. This says that $\varphi(x_0 T) = \varphi(T x_0)$ for all $T \in \mathcal{B}$ for all $x_0 \in \mathbb{C}\Gamma$.

We want to extend this property to all $\mathcal{B}(L^2)$. Notice also that $\varphi|_M = \tau$. If $x \in (L\Gamma)_1 = (M)_1$ is arbitrary, then let $x_n \in (\mathbb{C}\Gamma)_1$ such that $\|x - x_n\|_2 \to 0$ (by Kaplansky's density theorem). So

$$\begin{aligned}
\varphi(xT) &= \varphi((x - x_n)T) + \varphi(x_n T)\\
&= \varphi((x - x_n)T) + \varphi(T x_n)\\
&= \varphi((x - x_n)T) + \varphi(Tx) + \varphi(T(x_n - x))
\end{aligned}$$

By Cauchy-Schwarz, we have

$$\begin{aligned}
|\varphi((x_n - x)T)| &\leq \varphi((x_n - x)(x_n - x)^*)^{1/2} \varphi(T^*T)^{1/2}\\
&= \tau((x_n - x)(x_n - x)^*)^{1/2} \varphi(T^*T)^{1/2}\\
&= \|x - x_n\|_2 \varphi(T^*T)^{1/2}\\
&\xrightarrow{n\to\infty} 0.
\end{aligned}$$

We get a similar bound for $\varphi(T(x_n - x))$.

($\Longrightarrow$): If $L(\Gamma)$ is amenable and $\varphi \in S(\mathcal{B}(\ell^2(\Gamma)))$ is a hypertrace, then there is an embedding $\ell^\infty(\Gamma) \to \mathcal{B}(\ell^2(\Gamma))$ by $f \mapsto M_f$ on $\ell^2(\Gamma)$. Note that $u_g := \lambda(g)$ satisfies $u_g M_f u_g^* = M_{gf}$. So $\varphi(M_{gf}) = \varphi(u_g M_f u_g^*) = \varphi(M_f)$. In other words, if we define $f \mapsto M_f \mapsto \varphi(M_f)$, we get an invariant mean on Γ. $\square$

Remark 1.1. This is not the proof Murray and von Neumann gave to show that $L(S_\infty)$ and $L(\mathbb{F}_n)$ are non-isomorphic. And von Neumann was the one who formulated the definition of amenable groups!

1.2 Amenability and nonisomorphism of S_∞, $\mathbb{F}_2$, and $S_\infty \times \mathbb{F}_2$

Proposition 1.1. *$\mathbb{F}_2$ is not amenable.*

Proof. Assume φ is an invariant mean on $\ell^\infty(\mathbb{F}_2)$ (where $\mathbb{F}_2$ is the words in letters a and b). Take $A \subseteq \Gamma$ to be all words in a and b that start by a^n with $n \neq 0$. Then $\Gamma = S \cup aS \cup \{e\}$. This gives $1 = \varphi(\mathbb{1}_\Gamma) \leq 2\varphi(\mathbb{1}_S)$ On the other hand, $S, bS, b^{-1}S$ are disjoint. This gives that $3\varphi(\mathbb{1}_S) \leq \varphi(\mathbb{1}_\Gamma) = 1$. This is a contradiction. $\square$

Proposition 1.2. *S_∞ is amenable.*

Proof. It satisfies the Følner property. $\square$

Corollary 1.2. *$L(\mathbb{F}_n) \ncong L(S_\infty)$.*

Proof. The former is not amenable, while the latter is amenable. $\square$

Proposition 1.3. *If $\Gamma_0 \subseteq \Gamma$ and Γ is amenable, then Γ_0 is amenable.*

Proof. We have an embedding $\ell^\infty(\Gamma_0) \to \ell^\infty(\Gamma)$ as follows: take representatives g_n of the cosets in Γ/Γ_0. Then if $f \in \ell^\infty(\Gamma_0)$, we get $\widetilde{f} \in \ell^\infty(\Gamma)$ given by $\widetilde{f}(hg_n) = f(h)$ for all n. This is like reproducing $\ell^\infty(\Gamma_0)$ in $\ell^\infty(\Gamma)$ $|\Gamma/\Gamma_0|$-many times.

If $\varphi \in S(\ell^\infty(\Gamma))$ is an invariant mean, then $\varphi|_{\ell^\infty(\Gamma_0)}$ is an invariant for mean for Γ_0. $\square$

Corollary 1.3. *$\mathbb{F}_2 \times S_\infty$ is not amenable. Moreover, if $\Gamma \supseteq \mathbb{F}_2$, then Γ is not amenable.*

Corollary 1.4. *$L(S_\infty)$, $L(\mathbb{F}_2)$, and $L(S_\infty \times \mathbb{F}_2)$ are nonisomorphic.*

Proof. $L(S_\infty)$ is amenable, and the other two are not. $L(\mathbb{F}_2)$ does not have property Gamma, but $L(S_\infty \times \mathbb{F}_2)$ does have property Gamma. $\square$